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Cardinal Logarithmic Splines and Mellin Transform

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1. INTRODUCTION

Let J denote a non-empty open subinterval of the real line $\mathbb{R}$ and let $\mathfrak{k} = (x_n)_{n\in\mathbb{Z}}$ be a strictly monotone bi-infinite sequence of numbers $x_n \in J$. For each number $m \in \mathbb{N}^\times$ let $\mathfrak{S}_m(J; \mathfrak{k})$ denote the vector space over the field $\mathbb{C}$ of all complex cardinal spline functions of degree m on J with knot sequence $\mathfrak{k}$, i.e., the vector space of all complex-valued functions $s \in \mathscr{C}^{m-1}(J)$ such that the restriction of s to each compact subinterval $[x_n, x_{n+1}]$ $(n \in \mathbb{Z})$ of J is a polynomial function of degree $\leqslant m$ with complex coefficients.

It is well known that the simplest spline functions belonging to the vector space $\mathfrak{S}_m(J; \mathfrak{k})$ are the truncated power functions

$$s_{m,n}: J \ni x \rightsquigarrow (x - x_n)_+^m \qquad (n \in \mathbb{Z}), \tag{1}$$

where, as usual, for any mapping $f: J \to \mathbb{R}$, its positive part on J, $\max(0, f)$, is denoted by f_+. One reason for the importance of the spline functions $(s_{m,n})_{n\in\mathbb{Z}}$ is the fact that the basis splines $s \in \mathfrak{S}_m(J; \mathfrak{k})$ in the sense of Curry and Schoenberg [1] may be represented as (finite) linear combinations with real coefficients of the truncated power functions $(s_{m,n})_{n\in\mathbb{Z}}$. Indeed, for any given $n \in \mathbb{Z}$, we have

$$s = \sum_{0\leqslant k\leqslant m+1} c_k s_{m,n+k}; \tag{2}$$

and the conditions

$$\operatorname{Supp}(s) \subseteq [x_n, x_{n+m+1}]$$

("small support" condition), and

$$\int_{x_n}^{x_{n+m+1}} s(t)\, dt = 1$$

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(normalization) that characterize the basis spline $s \in \mathfrak{S}_m(J; \mathfrak{t})$ imply that its coefficients $(c_k)_{0 \leqslant k \leqslant m+1}$ are uniquely determined by the system of linear equations

$$\begin{aligned} \sum_{0 \leqslant k \leqslant m+1} c_k (x_{n+k})^l &= 0 \qquad &\text{for} \quad 0 \leqslant l \leqslant m \\ &= (-1)^{m+1} \cdot (m+1) \qquad &\text{for} \quad l = m+1. \end{aligned} \tag{3}$$

For further results concerning B-splines (which are numerically more satisfactory than the truncated power basis), the reader is referred to the recent book by de Boor [2].

The notion of truncated power function (1) is closely related to the Heaviside unit step function

$$\begin{aligned} Y: x &\rightsquigarrow 1 \qquad \text{if} \quad x \geqslant 0 \\ &\rightsquigarrow 0 \qquad \text{if} \quad x < 0, \end{aligned}$$

that plays an important rôle in Schwartz distribution theory. It is well known under the name "discontinuous factor" or "unit impulse" in some other branches of applied mathematical sciences. In particular, in integral transform analysis, mathematical physics, electrical engineering, and signal processing various different kinds of integral representations are used for these "factors." To be more specific, recall that the one-sided Laplace transform of the monomial function $\mathbb{R} \ni x \rightsquigarrow x^m$ $(m \in \mathbb{N}^\times)$ is given by $z \rightsquigarrow m!/z^{m+1}$ in the complex open right half-plane. Thus the Laplace inversion theorem implies that the truncated power functions (1) admit the integral representations

$$s_{m,n}: J \ni x \rightsquigarrow \frac{m!}{2\pi} \int_{c-i\infty}^{c+i\infty} \frac{e^{(x-x_n)z}}{z^{m+1}} \frac{dz}{i} \qquad (c \in \mathbb{R}_+^\times) \tag{4}$$

for all numbers $n \in \mathbb{Z}$. Needless to say, the choice of a specific integral representation should be appropriate to the problem dealt with. Appropriate integral transforms reveal themselves to be pliable and versatile tools.

The main idea of the present paper is to establish a suitable integral representation of the truncated power functions (1) by means of the inverse Mellin transform (Section 3). Although this integral transform is closely related to the Fourier transform and to the (two-sided) Laplace transform, it has its own peculiar uses. Its application to problems arising in various parts of analytic number theory and in the theory of difference equations (Nörlund [5]) is particularly effective. However, to the author's best knowledge the first application of the Mellin transform to problems arising in the theory of spline functions was discussed in his papers [9, 10].—In the present paper we will show that our approach, which is based on the Mellin inversion theorem as formulated in Section 2 and which differs totally from the original method developed by Newman and Schoenberg [4] for cardinal logarithmic spline functions (Section 4), makes it possible to establish the whole theory of these splines in a very lucid and economical manner. In particular, the asymptotic behaviour of the cardinal logarithmic spline functions when their degrees tend to infinity can be easily analyzed by this technique (Section 6). Thus the contour integral representation for the cardinal logarithmic splines (Theorem 4), which is the central result of the present paper, particularly complements the author's previous treatments [9, 10] of this topic.

2. Inversion of the Mellin Transform

Let I denote a non-empty subinterval of $\mathbb{R}$ and let g be a function holomorphic in the open strip $\Sigma_I = \{z \in \mathbb{C} \mid \operatorname{Re} z \in I\}$ such that

$$\lim_{|\operatorname{Im} z| \to +\infty} g(z) = 0$$

holds uniformly whenever $\operatorname{Re} z$ varies in any compact subset of the basis I of Σ_I. Then the following inversion theorem for the Mellin transform holds:

Theorem 1. *Suppose that the function g satisfies the aforementioned conditions. In addition, suppose that for each point $x_0 \in I$ the function*

$$\mathbb{R} \ni y \rightsquigarrow g(x_0 + iy) \in \mathbb{C}$$

belongs to the complex Lebesgue space $L^1(\mathbb{R})$. Then, for each $x \in \mathbb{R}_+^\times$ the integral

$$^t\mathscr{M}g(x) = \frac{1}{2\pi} \int_{c-i\infty}^{c+i\infty} g(z)\, x^{-z}\, \frac{dz}{i} \qquad (c \in I), \tag{5}$$

along a line in Σ_I parallel to the imaginary axis, is independent of the particular choice of the constant c in the basis I of Σ_I. For each $z \in \Sigma_I$, the Mellin transform of $^t\mathscr{M}g$, i.e., the line integral

$$\mathscr{M}(^t\mathscr{M}g)(z) = \int_{\mathbb{R}_+^\times} {}^t\mathscr{M}g(x)\, x^z\, \frac{dx}{x}, \tag{6}$$

exists and equals $g(z)$.

The proof follows via a displacement of the path of integration in (5) parallel to itself by an application of Cauchy's integral theorem. In fact, to within a change of variable, (6) is the Fourier inversion formula. For details the reader is referred to the classical treatise of Titchmarsh [12]. In this connection also see Oberhettinger [6].

3. Truncated Power Functions

Let $(\Gamma_m)_{m\in\mathbb{N}^\times}$ denote the sequence of partial products in the classical Gauss representation of the gamma function Γ, i.e., let Γ_m denote the meromorphic function

$$\Gamma_m : z \rightsquigarrow \frac{m!\, m^z}{\prod_{0\leqslant k\leqslant m}(z+k)} \qquad (m\in\mathbb{N}^\times). \tag{7}$$

Then, for each $z\in\mathbb{C}$ in the open right half-plane $\operatorname{Re} z>0$, successive performing of integration by parts on $t\rightsquigarrow(1-t/m)^m\, t^{z-1}$ yields the identity

$$\int_0^m \left(1-\frac{t}{m}\right)^m t^z\,\frac{dt}{t}=\Gamma_m(z) \qquad (\operatorname{Re} z>0) \tag{8}$$

for each number $m\in\mathbb{N}^\times$. Since the left hand side of (8) is the Mellin transform of the function $\mathbb{R}\ni t\rightsquigarrow(1-t/m)_+^m$ at the point $z\in\mathbb{C}$ with $\operatorname{Re} z>0$, the injectivity of the Mellin transform $\mathscr{M}$ implies via its inversion formula (Theorem 1) the following integral representation of the truncated power functions, a representation which is of interest in its own right.

THEOREM 2. *Let the numbers $m\in\mathbb{N}^\times$ and $t\in\mathbb{R}$ be fixed. Then we have*

$$\left(1-\frac{t}{m}\right)_+^m=\frac{1}{2\pi}\int_{c-i\infty}^{c+i\infty}\Gamma_m(z)\,t^{-z}\,\frac{dz}{i} \qquad (c>0), \tag{9}$$

where Γ_m is defined by (7) and the line integral is independent of the particular choice of $c>0$.

From (2) and (4) resp. (9) we may deduce two kinds of integral representations of the basis splines $s\in\mathfrak{S}_m(J;\mathfrak{t})$. The coefficients $(c_k)_{0\leqslant k\leqslant m+1}$ involved are well determined by (3). The first kind of integral representation will be exploited in an investigation of the cardinal spline interpolation at the sequence $\mathfrak{t}=\mathbb{Z}$ of equidistant knots on the whole real line by means of cardinal exponential splines in the sense of I. J. Schoenberg (cf. Section 7 infra). Another kind of integral representation of the basis splines using suitable closed rectifiable paths and rational integrands, has been pointed out by Meinardus [3]. However, the main concerns of the present paper are cardinal logarithmic splines in the sense of Newman and Schoenberg [4].

4. Cardinal Logarithmic Spline Functions

Let $J = \mathbb{R}_+^\times$ be the open positive real half-line and define the knot sequence $\mathfrak{t}_0 = (x_n)_{n \in \mathbb{Z}}$ according to

$$x_n = h_0^n \qquad (n \in \mathbb{Z}),$$

where the real number $h_0 > 1$ denotes a fixed step width. Consider the slightly modified logarithmic function

$$f_0 \colon \mathbb{R}_+^\times \ni x \rightsquigarrow \frac{\log x}{\log h_0} \in \mathbb{R}.$$

For each number $m \in \mathbb{N}^\times$ a real-valued function $S_m \in \mathfrak{S}_m(\mathbb{R}_+^\times; \mathfrak{t}_0)$ is called a cardinal logarithmic spline function of degree m with step width h_0 provided the following two conditions are satisfied:

(i) S_m satisfies the inhomogeneous linear geometric difference equation of the first order

$$S_m(h_0 x) - S_m(x) = 1$$

whenever $x \in J = \mathbb{R}_+^\times$.

(ii) S_m interpolates the function f_0 in the knot sequence $\mathfrak{t}_0$, i.e., the interpolation property

$$S_m(x_n) = f_0(x_n) = n$$

holds for all $n \in \mathbb{Z}$.

Given numbers $m \in \mathbb{N}^\times$ and $h_0 > 1$, there exists a unique cardinal logarithmic spline S_m of degree m with step width h_0. It can be constructed in the following way: Starting with the step function

$$\mathbb{R}_+^\times \ni x \rightsquigarrow S_m^{(m)}(x) = (-1)^{m-1}\, m! \sum_{n \in \mathbb{Z}} x_{-nm} Y(x_n - x),$$

calculate by successive integration (cf. Section 3) the functions

$$\mathbb{R}_+^\times \ni x \rightsquigarrow S_m^{(m-k)}(x) = -\int_x^\infty S_m^{(m-k+1)}(t)\, dt \qquad (1 \leqslant k \leqslant m-1).$$

Then, finally, the mth integration step

$$\mathbb{R}_+^\times \ni x \rightsquigarrow S_m(x) = \int_1^x S_m'(t)\, dt$$

furnishes for the desired cardinal logarithmic spline functions S_m the following explicit representation, which is valid uniformly for all points $x \in \mathbb{R}_+^\times$:

$$S_m(x) = \sum_{n \in \mathbb{Z}} ((1 - x_n)_+^m - (1 - xx_n)_+^m) \qquad (m \in \mathbb{N}^\times). \tag{10}$$

If formula (9) of Theorem 2 supra is inserted into (10) we can establish by an elementary computation the following:

THEOREM 3. *Let $(S_m)_{m \geqslant 1}$ be the sequence of cardinal logarithmic spline functions of degree m with step width $h_0 > 1$. Then the representation*

$$S_m(x) = \sum_{n \in \mathbb{Z}} \frac{1}{2\pi i} \int_{c - i\infty}^{c + i\infty} \Gamma_m(z)\, h_0^{-z(f_0(m) + n)} (1 - x^{-z})\, dz \tag{11}$$

holds uniformly for all points $x \in \mathbb{R}_+^\times$ and for each number $m \in \mathbb{N}^\times$. The integral along a line parallel to the imaginary axis in the complex open right half-plane $\operatorname{Re} z > 0$ *is independent of the particular choice of the real constant* $c > 0$.

The representaion formula (11) of S_m sharpens an asymptotic result of [9]. It will be the key to our contour integral representation formula (Theorem 4 infra) in the next section and to our discussion in Section 6 of the asymptotic behaviour of the sequence $(S_m)_{m \geqslant 1}$ as the degree m tends to infinity.

5. An Integral Representation of Cardinal Logarithmic Splines

Arguing as in note [9], let the infinite series (11) be divided into two parts such that the first one includes the summation over all numbers $n \in \mathbb{N}$, whereas the second one is concerned with the summation over all integers $n \leqslant -1$. In view of the uniform convergence with respect to the variable z of the first sum, we may change directly the order of integration and summation. However, such an interchange is not permissible in the second sum. If we observe that in the second sum the origin $z_0 = 0$ of $\mathbb{C}$ forms a removable singularity of the integrands, we may overcome the difficulty by translating the path of integration parallel to the imaginary axis to the open left half-plane. Taking into account that on the new contour the second sum converges uniformly with respect to z, the interchange of integration and summation is allowed. By putting the two parts together, the result may be described as follows. Let $c > 0$ and $d \in]-1, 0[$ be any fixed real numbers and let the straight lines parallel to the imaginary axis of $\mathbb{C}$

$$L_1 = \{z \in \mathbb{C} \mid \operatorname{Re} z = c\}, \qquad L_2 = \{z \in \mathbb{C} \mid \operatorname{Re} z = d\}$$

be endowed with a positive orientation in such a way that their juxtaposition

$$L = L_1 \vee L_2$$

in the one-point compactification of $\mathbb{C}$ admits the topological index (winding number)

$$\operatorname{Ind}_L(z_0) = 1$$

with respect to the origin $z_0 = 0$. Furthermore, introduce for each point $x \in \mathbb{R}_+^\times$ the sequence $(F_{m,x})_{m \geqslant 1}$ of complex-valued functions

$$F_{m,x}: z \rightsquigarrow \Gamma_m(z)\, h_0^{-zf_0(m)} \frac{1 - x^{-z}}{1 - h_0^{-z}} \qquad (m \in \mathbb{N}^\times). \tag{12}$$

Then we obtain, by Theorem 3 and the reasoning above, the central result of this paper.

THEOREM 4. *The cardinal logarithmic spline functions* $(S_m)_{m \geqslant 1}$ *admit the contour integral representation*

$$S_m(x) = \frac{1}{2\pi i} \int_L F_{m,x}(z)\, dz \qquad (m \in \mathbb{N}^\times), \tag{13}$$

which holds uniformly for all points $x \in \mathbb{R}_+^\times$. *The kernel* $F_{m,x}$ *is defined by* (12) *and* L *denotes the boundary of the vertical strip* $\Sigma_{]d,c[}$ *as indicated above.*

6. Asymptotic Analysis

The line integral (13) along the circuit L can easily be evaluated by an application of the calculus of residues. In fact, one verifies that for all numbers $m \in \mathbb{N}^\times$ the functions $F_{m,x}$ have simple poles located at the equidistant points

$$z_k = \frac{2\pi i k}{\log h_0} \qquad (k \in \mathbb{Z})$$

on the imaginary axis of $\mathbb{C}$ and that we have, for all $x \in \mathbb{R}_+^\times$,

$$\operatorname{Res}(F_{m,x}, z_0) = f_0(x),$$

$$\operatorname{Res}(F_{m,x}, z_k) = \frac{1}{\log h_0} \Gamma_m(z_k)(1 - x^{-z_k}) e^{-2\pi i k f_0(m)} \qquad (k \in \mathbb{Z}^\times),$$

$$\operatorname{Ind}_L(z_k) = 1 \qquad (k \in \mathbb{Z}).$$

At all the order poles $z \notin \{z_k \{ k \in \mathbb{Z}\}$ of the functions $F_{m,x}$ in the complex plane $\mathbb{C}$ we have $\operatorname{Ind}_L(z) = 0$. Thus we obtain from (13) by an application of Cauchy's residue theorem

$$\begin{aligned} S_m(x) &= \sum_{k \in \mathbb{Z}} \operatorname{Res}(F_{m,x}, z_k) \operatorname{Ind}_L(z_k) \\ &= f_0(x) + \frac{1}{\log h_0} \sum_{k \in \mathbb{Z}^\times} \Gamma_m(z_k)(1 - x^{-z_k}) e^{-2\pi i k f_0(m)}, \end{aligned} \tag{14}$$

uniformly for all points $x \in \mathbb{R}_+^\times$ and all numbers $m \in \mathbb{N}^\times$. We conclude from (14) by a standard denseness argument (cf. [11]) that $\lim_{m \to \infty} S_m(x) = f_0(x)$ holds if and only if the point $x \in \mathbb{R}_+^\times$ satisfies the condition

$$\sum_{k \in \mathbb{Z}^\times} \Gamma(z_k)(1 - x^{-z_k}) e^{2\pi i k t} = 0 \tag{15}$$

for all points $t \in [0, 1[$. If the identity (15) is combined with the injectivity of the Fourier transform, our main result of this section becomes obvious.

THEOREM 5. *The condition* $\lim_{m \to \infty} S_m(x) = f_0(x)$ *is satisfied at the points* $x \in \mathbb{R}_+^\times$ *if and only if x belongs to the knot sequence* $\mathfrak{t}_0$, *i.e., if and only if x coincides with one of the interpolation knots of the sequence* $(S_m)_{m \geqslant 1}$.

In other words, Theorem 5 states that the sequence $(S_m)_{m \geqslant 1}$ of cardinal logarithmic spline functions converges pointwise on $\mathbb{R}_+^\times$ towards f_0 as $m \to \infty$ only at those points $x \in \mathbb{R}_+^\times$ where the convergence holds trivially by the interpolation property (ii). This striking fact is called the Newman–Schoenberg phenomenon. Formula (14) shows that the bi-infinite sequence $(z_k)_{k \in \mathbb{Z}^\times}$ of simple poles $\neq 0$ located on the imaginary axis of $\mathbb{C}$ is responsible for the occurrence of the pointwise divergence phenomenon.

7. CONCLUDING REMARKS

The integral representation of the cardinal logarithmic splines, established in Theorem 4 via the inverse Mellin transform (Theorem 1), simplifies the asymptotic analysis that is given in Section 6 supra by the same method as in [9]. In contrast to this complex integral transform method, Refs. [7, 8] are based on a real integral transform and a suitable second-order refinement of Karamata's Abel–Tauber theorem. For a survey of the transform methods the reader is referred to Ref. [11], which will be published in near future. Furthermore, an application of the inverse Laplace transform to the cardinal exponential splines mentioned above will be outlined in a forthcoming paper. In this case too, the integral representation technique gives more insight into what actually happens when the degree of the cardinal exponental splines tends to infinity. In particular it shows that the pointwise asymptotic behaviours of the cardinal exponential spline interpolants and the cardinal logarithmic splines are totally different.

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